

Properties of Squeezed-State Excitations

Vladimir I. Man'ko

*Lebedev Physical Institute
Leninsky Prospekt 53, 117924 Moscow, Russia*

tel.: (095) 1326219, fax: (095) 9382251,

e-mail: manko@na.infn.it

and

Alfred Wünsche

*Arbeitsgruppe "Nichtklassische Strahlung"
der Max-Planck-Gesellschaft
Rudower Chaussee 5, 12489 Berlin, Germany*

tel.: (4930) 6392 3982, fax: (4930) 6392 3990,

e-mail: wuensche@photon.fta-berlin.de

Abstract

The photon distribution function of a discrete series of excitations of squeezed coherent states is given explicitly in terms of Hermite polynomials of two variables. The Wigner and the coherent-state quasiprobabilities are also presented in closed form through the Hermite polynomials and their limiting cases. Expectation values of photon numbers and their dispersion are calculated. Some three-dimensional plots of photon distributions for different squeezing parameters demonstrating oscillatory behaviour are given.

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1. Introduction

The photon and quadrature statistics of nonclassical states of light such as squeezed states [1, 2], even and odd coherent states [3, 4, 5], displaced Fock (or number) states [6, 7, 8, 9] or displaced and squeezed Fock states [10], and corresponding multimode states as, for example, two-mode squeezed vacuum [11] differs essentially from the statistics of light in coherent states [12, 13, 14, 15, 16] which has the Poissonian photon distribution and Gaussian quadrature statistics with equal minimal dispersions of the both noncorrelated quadratures. The distributions for nonclassical light have frequently oscillatory character [17, 18, 19]. For correlated light [20], this has been found in [21] and this phenomenon takes place for generalized correlated states [22] as well.

In present work, we concentrate on statistical properties of states studied in [23, 24] and the aim of the work is to obtain the explicit analytic expressions in terms of classical polynomials and in terms of multivariable Hermite polynomials for the Wigner quasiprobability, the coherent-state quasiprobability and the photon distribution function. The discussed states are related to general formulations of different types of nonclassical states given in [25, 26, 27, 28]. These states form a discrete series of excitations of squeezed coherent states in an analogous way as the displaced Fock states can be considered as discrete excitations of the coherent states. They were introduced for the diagonalization of a class of transition operators which make the transition from the density operator to the quasiprobabilities of the complete Gaussian class by forming the trace. For the formulation of orthogonality and completeness relations, it was convenient to define these states in a nonnormalized form with a parametrization in such a way that in the limit of vanishing squeezing parameter they become identical with the displaced Fock states. Other limiting cases of these states are the eigenstates of the canonical operators and a discrete series of their excitations which has been not considered up to now. The squeezed-state excitations are interesting because they combine and generalize the properties of well-known important states as squeezed states, number states, and coherent states. Such kind of states might appear in the process of parametric excitation of the quantum electromagnetic field in resonators with moving walls for which nonstationary Casimir effect [29, 30, 31, 32, 33] produces squeezing in quadrature components both in coherent states and Fock states.

We will calculate in the present paper the Wigner and the coherent-state quasiprobability and furthermore the photon distribution function in terms of the multivariable Hermite polynomials [34]. The distributions and quasidistributions were preliminarily

given in [24] in a more conventional form. It was shown that the multivariable Hermite polynomials are useful to describe the statistics and some other properties of nonclassical states [35, 36, 37, 38, 39]. One also could realize experimental creation of these states using methods of state engineering suggested in [40].

It should be remarked that there exists nonclassical light with different degree of the nonclassicality. For example, the photon statistics of the generalized coherent states [25, 26] and their partial cases like Schrödinger cats [4] is the same Poissonian statistics as for usual coherent states. But the photon quadrature statistics of the generalized coherent states is essentially different from the Gaussian statistics of the uncorrelated quadratures in the coherent states. Nonlinear Kerr effect and other nonlinear Hamiltonians depending only on photon number operator produce in the process of time evolution from the initial coherent states the generalized coherent states and never change the photon statistics but influence the quadrature statistics of the light, i.e. the quasiprobabilities. From that point of view, the superposition of coherent states discussed in [4] are closer to classical than Schrödinger cats of even and odd coherent states [3] in which both photon number and quadrature statistics differ from ones in the coherent states. As we will point out the quantum statistics of squeezed-states excitations is essentially nonclassical in both aspects, i.e. the photon statistics and the quadrature statistics are essentially different from these statistics in the classical coherent states. There may be different understandings of notions of classical light. Usually, classical light is considered as coherent-state light and the degree of deviation from the classical light may be evaluated as the minimal distance to coherent states [41] or, in another way, by the minimal parameter s of the s -ordered quasiprobabilities for which regions of negativity in the quasidistributions begin to appear [42].

The paper is organized as follows. In the next section, we consider the definitions of the squeezed-state excitations and calculate the normalization constants of these states in closed form. In section 3, we express the general products of the squeezed-state excitations in terms of multivariable Hermite polynomials. In section 4, we give the explicit expressions of the Wigner quasiprobability in terms of the multivariable Hermite polynomials and discuss some properties of the plots of the quasidistributions. In section 5, we calculate in closed form the photon distribution of the squeezed-state excitations and using plots of the distributions for some sets of parameters we demonstrate the oscillatory behaviour of the photon distribution. The mean value of the photon number is found explicitly and discussed in section 6 as well as dispersions of quadratures and uncertainty product. In two appendices A and B, the properties of the multivariable Hermite polynomials are reviewed and some new relations of these polynomials to classical polynomials like Legendre, Jacobi, and Gegenbauer polynomials which we used to discuss the photon statistics of squeezed-state excitations are derived.

2. Definition of a set of squeezed-state excitations and normalization

A discrete set of squeezed-state excitations $|\beta, n; \zeta\rangle$, ($n = 0, 1, 2, \dots$) has been introduced in [23, 24] for the purpose of a diagonal representation of the complete Gaussian class of quasiprobabilities. These states possess interesting properties of orthogonality and completeness. An appropriate way is to introduce these states in three steps. The first step is the introduction of squeezed vacuum states with the complex squeezing parameter ζ in the nonunitary approach [43, 44] and in a nonnormalized form according to

$$\begin{aligned} |0, 0; \zeta\rangle &\equiv (1 + |\zeta|^2)^{\frac{1}{4}} \exp\left(-\frac{\zeta}{2} a^{\dagger 2}\right) |0\rangle \\ &= (1 + |\zeta|^2)^{\frac{1}{4}} \sum_{m=0}^{\infty} \sqrt{\frac{(2m-1)!!}{2^m m!}} (-\zeta)^m |2m\rangle, \end{aligned} \quad (2.1)$$

with the following scalar product leading to a nonstandard normalization

$$\langle 0, 0; \xi | 0, 0; \zeta \rangle = \left(\frac{(1 + |\xi|^2)(1 + |\zeta|^2)}{(1 - \zeta \xi^*)^2} \right)^{\frac{1}{4}}, \quad \langle 0, 0; -\zeta | 0, 0; \zeta \rangle = 1. \quad (2.2)$$

The next step is the introduction of a discrete set of excitations of the squeezed vacuum states according to

$$|0, n; \zeta\rangle \equiv \frac{1}{\sqrt{n!}} \left(\frac{a^\dagger - \zeta^* a}{\sqrt{1 + |\zeta|^2}} \right)^n |0, 0; \zeta\rangle, \quad (n = 0, 1, 2, \dots). \quad (2.3)$$

The last step is the displacement of these states with the unitary displacement operator $D(\beta, \beta^*)$ according to

$$|\beta, n; \zeta\rangle \equiv D(\beta, \beta^*) |0, n; \zeta\rangle, \quad D(\beta, \beta^*) \equiv \exp(\beta a^\dagger - \beta^* a). \quad (2.4)$$

The states with arbitrary pairs of squeezing parameters ζ and $-\zeta$ and arbitrary displacement parameters β are mutually orthonormalized and obey a completeness relation

as follows

$$\langle \beta, m; -\zeta | \beta, n; \zeta \rangle = \delta_{m,n}, \quad \sum_{n=0}^{\infty} |\beta, n; \zeta \rangle \langle \beta, n; -\zeta| = I, \quad (2.5)$$

where I denotes the unity operator of the Fock space. For vanishing squeezing parameter, one obtains from $|\beta, n; \zeta \rangle$ the displaced Fock states $|\beta, n \rangle$

$$|\beta, n; 0 \rangle \equiv D(\beta, \beta^*) \frac{a^\dagger{}^n}{\sqrt{n!}} |0 \rangle = D(\beta, \beta^*) |n \rangle \equiv |\beta, n \rangle. \quad (2.6)$$

The relations (2.5) form the background for the introduction of the states $|\beta, n; \zeta \rangle$ in the described form with a nonstandard normalization.

The coordinate representation of the states $|\beta, n; \zeta \rangle$ has the form [24] ($\psi(q; \beta, n; \zeta) \equiv \langle q | \beta, n; \zeta \rangle$)

$$\begin{aligned} & \psi(q; \beta, n; \zeta) \\ &= \frac{1}{\sqrt{2^n n!}} \left(\sqrt{\frac{1 + \zeta^*}{1 - \zeta}} \right)^n H_n \left(\sqrt{\frac{1 + |\zeta|^2}{(1 - \zeta)(1 + \zeta^*)\hbar}} \left(q - \sqrt{\frac{\hbar}{2}} (\beta + \beta^*) \right) \right) \\ & \quad \left(\frac{1 + |\zeta|^2}{(1 - \zeta)^2 \hbar \pi} \right)^{\frac{1}{4}} \exp \left\{ -\frac{1 + \zeta}{1 - \zeta} \frac{1}{2\hbar} \left(q - \sqrt{\frac{\hbar}{2}} (\beta + \beta^*) \right)^2 + \frac{(\beta - \beta^*)q}{\sqrt{2\hbar}} - \frac{\beta^2 - \beta^{*2}}{4} \right\}, \end{aligned} \quad (2.7)$$

where $H_n(z)$ denotes the Hermite polynomials. This formula corresponds to the following connection between the boson operators a and a^\dagger and the canonical operators Q and P

$$a \equiv \frac{Q + iP}{\sqrt{2\hbar}}, \quad a^\dagger \equiv \frac{Q - iP}{\sqrt{2\hbar}}, \quad i\hbar[a, a^\dagger] = [Q, P] = i\hbar I, \quad (2.8)$$

and can be calculated using $\langle q | Q = q \langle q |$ and $\langle q | P = -i\hbar \frac{\partial}{\partial q} \langle q |$ and the normalization $\langle q | q' \rangle = \delta(q - q')$. The momentum representation of the states $|\beta, n; \zeta \rangle$ will be given in section 3.

In this section, we recalculate the normalization constant $N_n(|\zeta|)$ of the states $|\beta, n; \zeta \rangle$ given in [24] in form of the series

$$\begin{aligned} N_n^{-2}(|\zeta|) &\equiv \langle 0, n; \zeta | 0, n; \zeta \rangle \\ &= \left(\frac{1 + |\zeta|^2}{1 - |\zeta|^2} \right)^{n + \frac{1}{2}} \sum_{k=0}^{[n/2]} \frac{n!}{k!^2 (n - 2k)!} \left(\frac{|\zeta|}{1 + |\zeta|^2} \right)^{2k}. \end{aligned} \quad (2.9)$$

This series may be either reorganized or recalculated in the form of the integral

$$N_n^{-2}(|\zeta|) = \int_{-\infty}^{+\infty} dq |\psi(q; \beta, n; \zeta)|^2, \quad (2.10)$$

which is expressed in terms of Jacobi polynomials $P_n^{(j,k)}(z)$ or Legendre polynomials $P_n(z)$ as follows (see Appendix A)

$$N_n^{-2}(|\zeta|) = \sqrt{\frac{1+|\zeta|^2}{1-|\zeta|^2}} P_n^{(0,0)}\left(\frac{1+|\zeta|^2}{1-|\zeta|^2}\right) = \sqrt{\frac{1+|\zeta|^2}{1-|\zeta|^2}} P_n\left(\frac{1+|\zeta|^2}{1-|\zeta|^2}\right). \quad (2.11)$$

A representation by special Gegenbauer polynomials $C_n^{\frac{1}{2}}(z)$ with upper index $1/2$ is also possible. Thus, for the normalized states

$$|\beta, n; \zeta\rangle_{norm} \equiv N_n(|\zeta|)|\beta, n; \zeta\rangle, \quad (2.12)$$

one obtains in coordinate representation

$$\begin{aligned} \psi_{norm}(q; \beta, n; \zeta) &\equiv N_n(|\zeta|)\psi(q; \beta, n; \zeta) \\ &= \left(\frac{1-|\zeta|^2}{1+|\zeta|^2}\right)^{\frac{1}{4}} \left\{ P_n\left(\frac{1+|\zeta|^2}{1-|\zeta|^2}\right) \right\}^{-\frac{1}{2}} \psi(q; \beta, n; \zeta), \end{aligned} \quad (2.13)$$

where $\psi(q; \beta, n; \zeta)$ can be taken from Eq.(2.7). The normalization constants do not depend on the displacement parameter β since the unitary displacement operator is cancelled when forming the scalar products as can be seen from Eq.(2.4). The normalization with the normalization constant $N_n(|\zeta|)$ is only possible for $|\zeta| < 1$. However, the states $|\beta, n; \zeta\rangle$ are well defined also for $|\zeta| \geq 1$ and can be used for auxiliary purposes as, for example, for the formulation of completeness relations [23, 24, 43].

3. Calculation of the general scalar products

In this section, we calculate the general scalar product of two arbitrary states $|\alpha, m; \xi\rangle$ and $|\beta, n; \zeta\rangle$. Using Eq.(2.4) in connection with the product of two arbitrary displacement operators one can reduce this scalar product to the form

$$\begin{aligned}\langle\alpha, m; \xi|\beta, n; \zeta\rangle &\equiv \langle 0, m; \xi|(D(\alpha, \alpha^*))^\dagger D(\beta, \beta^*)|0, n; \zeta\rangle \\ &= \exp\left\{\frac{1}{2}(\alpha^*\beta - \alpha\beta^*)\right\}\langle 0, m; \xi|D(\beta - \alpha, \beta^* - \alpha^*)|0, n; \zeta\rangle \\ &\equiv \exp\left\{\frac{1}{2}(\alpha^*\beta - \alpha\beta^*)\right\}\langle 0, m; \xi|\beta - \alpha, n; \zeta\rangle.\end{aligned}\quad (3.1)$$

This formula shows that the general scalar product $\langle\alpha, m; \xi|\beta, n; \zeta\rangle$ can be obtained from the special scalar product $\langle 0, n; \xi|\beta, n; \zeta\rangle$ by the substitution $\beta \rightarrow \beta - \alpha$ and multiplication of the result of this substitution by the phase factor $\exp\{1/2(\alpha^*\beta - \alpha\beta^*)\}$.

According to Eq.(2.7) one can calculate the reduced scalar product by evaluating the following integral in coordinate representation

$$\begin{aligned}&\langle 0, m; \xi|\beta, n; \zeta\rangle \\ &= \frac{1}{\sqrt{2^{m+n}m!n!}}\left(\frac{1+\xi}{1-\xi^*}\right)^{\frac{m}{2}}\left(\frac{1+\zeta^*}{1-\zeta}\right)^{\frac{n}{2}}\frac{((1+|\xi|^2)(1+|\zeta|^2))^{\frac{1}{4}}}{\sqrt{(1-\zeta)(1-\xi^*)\hbar\pi}} \\ &\quad \exp\left\{-\frac{(\beta+\zeta\beta^*)(\beta^*+\xi^*\beta)}{2(1-\zeta\xi^*)}\right\}A,\end{aligned}\quad (3.2)$$

with the following abbreviation for an integral of the Hermite–Gauss type

$$\begin{aligned}A &\equiv \int_{-\infty}^{+\infty} dq H_m\left(\sqrt{\frac{1+|\xi|^2}{(1+\xi)(1-\xi^*)\hbar}} q\right) H_n\left(\sqrt{\frac{1+|\zeta|^2}{(1-\zeta)(1+\zeta^*)\hbar}}\left(q - \sqrt{\frac{\hbar}{2}}(\beta + \beta^*)\right)\right) \\ &\quad \exp\left\{-\frac{1-\zeta\xi^*}{(1-\zeta)(1-\xi^*)\hbar}\left(q - \sqrt{\frac{\hbar}{2}}\frac{1-\xi^*}{1-\zeta\xi^*}(\beta + \zeta\beta^*)\right)^2\right\}.\end{aligned}\quad (3.3)$$

With the substitution

$$x = \sqrt{\frac{1+|\xi|^2}{(1+\xi)(1-\xi^*)\hbar}} q, \quad (3.4)$$

one obtains an integral of the type of Eq.(B.10) in Appendix B. However, we wrote the exponential term in the integrand as a Gaussian with displaced argument with the consequence that the result of the integration is a pure polynomial. It can be expressed by the two-variable Hermite polynomials as follows

$$\begin{aligned}
A &= \sqrt{\frac{(1-\zeta)(1-\xi^*)\hbar\pi}{1-\zeta\xi^*}} H_{mn}^{\{R\}}(y_1, y_2), \\
R &= \frac{2}{1-\zeta\xi^*} \\
&\left(\begin{array}{cc} \frac{(1-\xi^*)(\xi+\zeta)}{1+\xi} & , -\sqrt{\frac{(1-\zeta)(1-\xi^*)(1+|\zeta|^2)(1+|\xi|^2)}{(1+\zeta^*)(1+\xi)}} \\ -\sqrt{\frac{(1-\zeta)(1-\xi^*)(1+|\zeta|^2)(1+|\xi|^2)}{(1+\zeta^*)(1+\xi)}} & , \frac{(1-\zeta)(\xi^*+\zeta^*)}{1+\zeta^*} \end{array} \right), \\
y_1 &= \sqrt{\frac{(1+\xi)(1+|\xi|^2)}{2(1-\xi^*)}} \frac{\beta^* - \zeta^*\beta}{1-\xi\zeta^*}, \quad y_2 = -\sqrt{\frac{(1+\zeta^*)(1+|\zeta|^2)}{2(1-\zeta)}} \frac{\beta - \xi\beta^*}{1-\xi\zeta^*}.
\end{aligned} \tag{3.5}$$

This can be written in terms of the usual Hermite polynomials with the following final result for the special scalar product in Eq.(3.2)

$$\begin{aligned}
&\langle 0, m; \xi | \beta, n; \zeta \rangle \\
&= \left(\frac{(1+|\xi|^2)(1+|\zeta|^2)}{(1-\zeta\xi^*)^2} \right)^{\frac{1}{4}} \exp \left\{ -\frac{(\beta + \zeta\beta^*)(\beta^* + \xi^*\beta)}{2(1-\zeta\xi^*)} \right\} \\
&\quad \frac{(-1)^n}{\sqrt{2^{m+n}m!n!}} \left(\sqrt{\frac{\xi + \zeta}{1-\zeta\xi^*}} \right)^m \left(\sqrt{\frac{\xi^* + \zeta^*}{1-\zeta\xi^*}} \right)^n \\
&\quad \sum_{j=0}^{\{m,n\}} \frac{(-1)^j m!n!}{j!(m-j)!(n-j)!} \left(\frac{2\sqrt{(1+|\xi|^2)(1+|\zeta|^2)}}{|\xi + \zeta|} \right)^j \\
&\quad H_{m-j} \left(\sqrt{\frac{1+|\xi|^2}{2(1-\zeta\xi^*)(\xi + \zeta)}} (\beta + \zeta\beta^*) \right) H_{n-j} \left(\sqrt{\frac{1+|\zeta|^2}{2(1-\zeta\xi^*)(\xi^* + \zeta^*)}} (\beta^* + \xi^*\beta) \right).
\end{aligned} \tag{3.6}$$

Recall that the general scalar product $\langle \alpha, m; \xi | \beta, n; \zeta \rangle$ can be obtained from this special scalar product $\langle 0, m; \xi | \beta, n; \zeta \rangle$ by the above mentioned simple substitutions (see Eq.(3.1)).

Let us consider special cases of the general scalar product. The special case $\xi = \zeta$ is important for the calculation of expectation values for the states $|\beta, n; \zeta\rangle$ (section 6). In this case, one finds from Eq.(3.6)

$$\langle 0, m; \zeta | \beta, n; \zeta \rangle$$

$$\begin{aligned}
&= \sqrt{\frac{1+|\zeta|^2}{1-|\zeta|^2}} \exp \left\{ -\frac{|\beta + \zeta\beta^*|^2}{2(1-|\zeta|^2)} \right\} \\
&\quad \frac{(-1)^n}{\sqrt{m!n!}} \left(\sqrt{\frac{\zeta}{1-|\zeta|^2}} \right)^m \left(\sqrt{\frac{\zeta^*}{1-|\zeta|^2}} \right)^n \sum_{j=0}^{\{m,n\}} \frac{(-1)^j m!n!}{j!(m-j)!(n-j)!} \left(\frac{1+|\zeta|^2}{|\zeta|} \right)^j \\
&\quad H_{m-j} \left(\sqrt{\frac{1+|\zeta|^2}{(1-|\zeta|^2)\zeta}} \frac{\beta + \zeta\beta^*}{2} \right) H_{n-j} \left(\sqrt{\frac{1+|\zeta|^2}{(1-|\zeta|^2)\zeta^*}} \frac{\beta^* + \zeta^*\beta}{2} \right). \tag{3.7}
\end{aligned}$$

In case of equal displacement parameter $\alpha = \beta$, the scalar product $\langle \beta, m; \xi | \beta, n; \zeta \rangle$ becomes independent on β and can be obtained by setting $\beta = 0$ in Eq.(3.6), i.e., by using the values of the Hermite polynomials for vanishing argument [23, 24] ($H_n(0) = \sum_{l=0}^{\infty} (-1)^l (2l)!/l! \delta_{n,2l}$)

$$\begin{aligned}
&\langle \beta, m; \xi | \beta, n; \zeta \rangle \\
&= \frac{1}{\sqrt{m!n!}} \left(\sqrt{\frac{1+|\xi|^2}{1-\zeta\xi^*}} \right)^{m+\frac{1}{2}} \left(\sqrt{\frac{1+|\zeta|^2}{1-\xi\xi^*}} \right)^{n+\frac{1}{2}} \\
&\quad \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^{k+l} m!n!}{k!l! \sqrt{(m-2k)!(n-2l)!}} \left(\frac{\xi + \zeta}{2(1+|\xi|^2)} \right)^k \left(\frac{\xi^* + \zeta^*}{2(1+|\zeta|^2)} \right)^l \delta_{m-2k, n-2l}. \tag{3.8}
\end{aligned}$$

This scalar product is nonvanishing only if $|m - n|$ is an even number. By setting $m = n + 2j$ and $m = n + 1 + 2j$, one obtains from Eq.(3.7)

$$\begin{aligned}
&\langle \beta, n + 2j; \xi | \beta, n; \zeta \rangle \\
&= \left(\frac{\sqrt{(1+|\xi|^2)(1+|\zeta|^2)}}{1-\zeta\xi^*} \right)^{n+\frac{1}{2}} \left(-\frac{\xi + \zeta}{2(1+|\xi|^2)} \right)^j \frac{\sqrt{(n+2j)!n!}}{(n+j)!} \\
&\quad \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(n+j)!}{l!(l+j)!(n-2l)!} \left(\frac{|\xi + \zeta|^2}{4(1+|\xi|^2)(1+|\zeta|^2)} \right)^l \\
&= \left(\frac{(1+|\xi|^2)(1+|\zeta|^2)}{(1-\zeta\xi^*)^2} \right)^{\frac{1}{4}} \left(\sqrt{\frac{1-\xi\xi^*}{1-\zeta\xi^*}} \right)^n \\
&\quad \left(-\frac{\xi + \zeta}{2(1+|\xi|^2)} \right)^j \frac{\sqrt{(n+2j)!n!}}{(n+j)!} P_n^{(j,j)} \left(\frac{\sqrt{(1+|\xi|^2)(1+|\zeta|^2)}}{|1-\zeta\xi^*|} \right), \tag{3.9}
\end{aligned}$$

and

$$\langle \beta, n + 1 + 2j; \xi | \beta, n; \zeta \rangle = 0, \tag{3.10}$$

where we have used the Jacobi polynomials $P_n^{(j,k)}(z)$ with $j = k$ for the representation (see Appendices A and B). In case of $\xi = \zeta$, one finds from Eqs.(3.9) and (3.10) or

from (3.7) by setting $\beta = 0$

$$\begin{aligned} & \langle \beta, n+2j; \zeta | \beta, n; \zeta \rangle \\ &= \sqrt{\frac{1+|\zeta|^2}{1-|\zeta|^2}} \left(-\frac{\zeta^*}{1-|\zeta|^2} \right)^j \frac{\sqrt{(n+2j)!n!}}{(n+j)!} P_n^{(j,j)} \left(\frac{1+|\zeta|^2}{1-|\zeta|^2} \right) \end{aligned} \quad (3.11)$$

and

$$\langle \beta, n+1+2j; \zeta | \beta, n; \zeta \rangle = 0. \quad (3.12)$$

The special case $j = 0$ of this scalar product gives the inverse squared normalization constant of the states $|\beta, n, \zeta\rangle$ which is given in Eqs.(2.9) and (2.11). In case of $\xi = -\zeta$, one finds from Eqs.(3.9) and (3.10) the orthogonality relations

$$\langle \beta, m; -\zeta | \beta, n; \zeta \rangle = \delta_{m,n}, \quad (3.13)$$

which are the main reason for the choice of the nonstandard normalization of the states $|\beta, n; \zeta\rangle$.

The states $|\alpha, 0; 0\rangle$ are identical with the coherent states $|\alpha\rangle$. Therefore, one obtains the Bargmann representation of the states $|\beta, n; \zeta\rangle$ by an analytic function $f(\alpha^*)$ as the following special case of the general scalar product [24]

$$\begin{aligned} f(\alpha^*) &\equiv \exp\left(\frac{|\alpha|^2}{2}\right) \langle \alpha, 0; 0 | \beta, n; \zeta \rangle \\ &= (1+|\zeta|^2)^{\frac{1}{4}} \exp\left\{ \alpha^* \beta - \frac{\zeta}{2} (\alpha^* - \beta^*)^2 - \frac{|\beta|^2}{2} \right\} \\ &\quad \frac{(\sqrt{\zeta^*})^n}{\sqrt{2^n n!}} H_n \left(\sqrt{\frac{1+|\zeta|^2}{2\zeta^*}} (\alpha^* - \beta^*) \right). \end{aligned} \quad (3.14)$$

From this relation, one easily finds the coherent-state quasiprobability of the states $|\beta, n; \zeta\rangle$. We give it only for $\beta = 0$ because the transition to arbitrary β can be made from $Q(\alpha, \alpha^*)$ for $\beta = 0$ by the simple substitutions $\alpha \rightarrow \alpha - \beta$ and $\alpha^* \rightarrow \alpha^* - \beta^*$. Thus, one obtains for the states $|0, n; \zeta\rangle_{\text{norm}}$ taking into account $|\alpha, 0; 0\rangle \equiv |\alpha\rangle$

$$\begin{aligned} Q(\alpha, \alpha^*) &\equiv \frac{1}{\pi} \frac{\langle \alpha | 0, n; \zeta \rangle \langle 0, n; \zeta | \alpha \rangle}{\langle 0, n; \zeta | 0, n; \zeta \rangle} \\ &= \frac{\sqrt{1-|\zeta|^2}}{\pi} \exp\left\{ -\left(\alpha \alpha^* + \frac{\zeta^*}{2} \alpha^2 + \frac{\zeta}{2} \alpha^{*2} \right) \right\} \\ &\quad \frac{|\zeta|^n}{2^n n! P_n\left(\frac{1+|\zeta|^2}{1-|\zeta|^2}\right)} H_n\left(\sqrt{\frac{1+|\zeta|^2}{2\zeta}} \alpha\right) H_n\left(\sqrt{\frac{1+|\zeta|^2}{2\zeta^*}} \alpha^*\right). \end{aligned} \quad (3.15)$$

Figure 1 shows in representation by the canonical variables (q, p) and with $\hbar = 1$ the coherent-state quasiprobability $Q(q, p)$ of the states $|0, n; \zeta\rangle_{\text{norm}}$ for the first six values

$n = 0, 1, \dots, 5$ and for the squeezing parameter $\zeta = (3 - \sqrt{5})/2 = 0.381966$ from a bird's perspective. The plot range is chosen as the maximal range from 0 to $1/(2\pi)$ which can be taken on by the normalized coherent-state quasiprobability $Q(q, p)$. The maximal height $1/(2\pi)$ is only reached for coherent states and the square root of the difference of the height of $Q(q, p)$ to $1/(2\pi)$ for the considered state is a measure for its distance to a "classical" state [41]. Thus, one can see the nonclassicality of states in such kind of figures in a very visual way. If one changes the sign of the squeezing parameter ζ , in our case by transition to $\zeta = -0.381966$, one obtains a quasiprobability $Q(q, p)$ with squeezing axes which are rotated about an angle $\pi/2$ relative to the primary squeezing axes. The seemingly strange value of the squeezing parameter $\zeta = (3 - \sqrt{5})/2$ was chosen for reason that the asymmetry measure $\overline{(\Delta N)^3}$ of the photon number distribution for squeezed coherent states takes on the maximal possible negative value that is not very relevant for the purposes of this paper. The qualitative features of the pictures for the quasidistributions in the central range of the squeezing parameter do not dramatically depend on the value of the squeezing parameter.

The position and momentum representation of the states $|\beta, n; \zeta\rangle$ can be obtained from the general scalar product by using that the states $|\beta, n; \zeta\rangle$ comprise the states $|q\rangle$ and $|p\rangle$ as the following special cases [23, 24]

$$\left| \frac{q + ip}{\sqrt{2\hbar}}, 0; 1 \right\rangle = (2\hbar\pi)^{\frac{1}{4}} \exp\left(i\frac{pq}{2\hbar}\right) |q\rangle, \quad \left| \frac{q + ip}{2\hbar}, 0; -1 \right\rangle = (2\hbar\pi)^{\frac{1}{4}} \exp\left(-i\frac{pq}{2\hbar}\right) |p\rangle. \quad (3.16)$$

In this way, one gets back the position representation of the states $|\beta, n; \zeta\rangle$ given in Eq.(2.7) which was the starting point of our calculations and in an analogous way we get the following momentum representation ($\psi(p; \beta, n; \zeta) \equiv \langle p | \beta, n; \zeta \rangle$)

$$\begin{aligned} & \psi(p; \beta, n; \zeta) \\ &= \frac{(-i)^n}{\sqrt{2^n n!}} \left(\sqrt{\frac{1 - \zeta^*}{1 + \zeta}} \right)^n H_n \left(\sqrt{\frac{1 + |\zeta|^2}{(1 + \zeta)(1 - \zeta^*)\hbar}} \left(p + i\sqrt{\frac{\hbar}{2}}(\beta - \beta^*) \right) \right) \\ & \quad \left(\frac{1 + |\zeta|^2}{(1 + \zeta)^2 \hbar \pi} \right)^{\frac{1}{4}} \exp \left\{ -\frac{1 - \zeta}{1 + \zeta} \frac{1}{2\hbar} \left(p + i\sqrt{\frac{\hbar}{2}}(\beta - \beta^*) \right)^2 - i\frac{(\beta + \beta^*)p}{\sqrt{2\hbar}} + \frac{\beta^2 - \beta^{*2}}{4} \right\}. \end{aligned} \quad (3.17)$$

The displaced Fock states $|\beta, n\rangle$ are the special cases of vanishing squeezing parameter ζ of the states $|\beta, n; \zeta\rangle$, i.e.

$$|\beta, n; 0\rangle = D(\beta, \beta^*) \frac{a^{\dagger n}}{\sqrt{n!}} |0, 0; 0\rangle \equiv |\beta, n\rangle, \quad (3.18)$$

and, therefore, one obtains the scalar products of displaced Fock states as special cases of the general scalar products of the states $|\beta, n; \zeta\rangle$ (see section 5).

4. Wigner quasiprobability

It is worth to describe the states $|\beta, n; \zeta\rangle$ in terms of quasiprobabilities as the Wigner quasiprobability [45] or the coherent-state quasiprobability [46, 47]. These quasiprobabilities have been calculated and given without detailed derivations in [24]. In particular, the Wigner quasiprobability was found there in terms of a finite series including products of Hermite polynomials. It turns that the series is summed up, i.e. it is possible to express the Wigner quasiprobability using the definition

$$W(q, p) = \frac{1}{2\hbar\pi} \int_{-\infty}^{+\infty} dx \exp\left(i\frac{px}{\hbar}\right) \psi\left(q - \frac{x}{2}\right) \psi^*\left(q + \frac{x}{2}\right), \quad (4.1)$$

by two-variable Hermite polynomials since the integral in Eq.(4.1) is for the considered states of the same form as calculated in Appendix B.

The Wigner quasiprobability for the states $|\beta, n; \zeta\rangle$ can be obtained from the Wigner quasiprobability for the states $|0, n; \zeta\rangle$ by a displacement β of the distribution in the complex plane of the complex variable α that means for the real variables (q, p) by the substitutions

$$\alpha \equiv \frac{q + ip}{\sqrt{2\hbar}} \rightarrow \alpha - \beta, \quad q \rightarrow q - \sqrt{\frac{\hbar}{2}}(\beta + \beta^*), \quad p \rightarrow p + i\sqrt{\frac{\hbar}{2}}(\beta - \beta^*). \quad (4.2)$$

Therefore, we restrict us to the calculation of the Wigner quasiprobability for the normalized states $|0, n; \zeta\rangle_{norm}$. From Eq.(4.1) in view of Eqs.(2.7) and (2.12), one finds

$$W(q, p) = \frac{N_n^2(|\zeta|)}{2^n n!} \left(\frac{|1 + \zeta|}{|1 - \zeta|}\right)^n \sqrt{\frac{1 + |\zeta|^2}{|1 - \zeta|^2 \hbar \pi}} \exp\left\{-\frac{|(1 + \zeta)q + i(1 - \zeta)p|^2}{(1 - |\zeta|^2)\hbar}\right\} \frac{1}{2\hbar\pi} B, \quad (4.3)$$

with the following abbreviation for an integral of the Hermite–Gauss type

$$B \equiv \int_{-\infty}^{+\infty} dx H_n\left(\sqrt{\frac{1 + |\zeta|^2}{(1 - \zeta)(1 + \zeta^*)\hbar}}\left(q - \frac{x}{2}\right)\right) H_n\left(\sqrt{\frac{1 + |\zeta|^2}{(1 + \zeta)(1 - \zeta^*)\hbar}}\left(q + \frac{x}{2}\right)\right) \exp\left\{-\frac{1 - |\zeta|^2}{|1 - \zeta|^2 \hbar}\left(\frac{x}{2} - \frac{(\zeta - \zeta^*)q + i|1 - \zeta|^2 p}{1 - |\zeta|^2}\right)^2\right\}. \quad (4.4)$$

The exponential function in the integrand is here complemented in a way that it is a displaced Gaussian function and the result of the integration is a pure polynomial. With the substitution

$$z = \sqrt{\frac{1 + |\zeta|^2}{(1 + \zeta)(1 - \zeta^*)\hbar}} \left(q + \frac{x}{2} \right), \quad (4.5)$$

one has an integral of the type in Eq.(B.10) in Appendix B. The result in terms of two-variable Hermite polynomials is

$$\begin{aligned} B &= (-1)^n |1 - \zeta| \sqrt{\frac{\hbar\pi}{1 - |\zeta|^2}} H_{nn}^{\{R\}}(y_1, y_2), \\ R &= \frac{2}{1 - |\zeta|^2} \begin{pmatrix} \frac{2\zeta(1 - \zeta^*)}{(1 + \zeta)} & , & -\frac{|1 - \zeta|(1 + |\zeta|^2)}{|1 + \zeta|} \\ -\frac{|1 - \zeta|(1 + |\zeta|^2)}{|1 - \zeta|} & , & \frac{2\zeta^*(1 - \zeta)}{(1 + \zeta^*)} \end{pmatrix}, \\ y_1 &= \sqrt{\frac{(1 + \zeta)(1 + |\zeta|^2)}{(1 - \zeta^*)\hbar}} \frac{(1 - \zeta^*)q - i(1 + \zeta^*)p}{1 - |\zeta|^2}, \\ y_2 &= -\sqrt{\frac{(1 + \zeta^*)(1 + |\zeta|^2)}{(1 - \zeta)\hbar}} \frac{(1 - \zeta)q + i(1 + \zeta)p}{1 - |\zeta|^2}. \end{aligned} \quad (4.6)$$

Thus, expression (4.3) contains no series but only well-known special functions both for the normalization constant and for the quadrature-dependent factor. This can be expressed also in terms of the series for the usual Hermite polynomials by

$$\begin{aligned} W(q, p) &= \frac{1}{\hbar\pi} \exp \left\{ -\frac{|(1 + \zeta)q + i(1 - \zeta)p|^2}{(1 - |\zeta|^2)\hbar} \right\} \\ &\quad \frac{\left(-\frac{1 + |\zeta|^2}{1 - |\zeta|^2} \right)^n}{P_n \left(\frac{1 + |\zeta|^2}{1 - |\zeta|^2} \right)} \sum_{j=0}^n \frac{(-1)^j n!}{j!^2 (n - j)!} \left(\frac{|\zeta|}{1 + |\zeta|^2} \right)^j \\ &\quad \left| H_j \left(\sqrt{\frac{1 + |\zeta|^2}{2\zeta(1 - |\zeta|^2)\hbar}} ((1 + \zeta)q + i(1 - \zeta)p) \right) \right|^2, \end{aligned} \quad (4.7)$$

or in complex representation

$$\begin{aligned} W(\alpha, \alpha^*) &= \frac{2}{\pi} \exp \left\{ -\frac{(\alpha + \zeta\alpha^*)(\alpha^* + \zeta^*\alpha)}{1 - |\zeta|^2} \right\} \\ &\quad \frac{\left(-\frac{1 + |\zeta|^2}{1 - |\zeta|^2} \right)^n}{P_n \left(\frac{1 + |\zeta|^2}{1 - |\zeta|^2} \right)} \sum_{j=0}^n \frac{(-1)^j n!}{j!^2 (n - j)!} \left(\frac{|\zeta|}{1 + |\zeta|^2} \right)^j \end{aligned}$$

$$H_j \left(\sqrt{\frac{1 + |\zeta|^2}{\zeta(1 - |\zeta|^2)}} (\alpha + \zeta \alpha^*) \right) H_j \left(\sqrt{\frac{1 + |\zeta|^2}{\zeta^*(1 - |\zeta|^2)}} (\alpha^* + \zeta^* \alpha) \right). \quad (4.8)$$

In representation by the canonical variables (q, p) and with $\hbar = 1$, Fig.2 shows the Wigner quasiprobability $W(q, p)$ of the states $|0, n; \zeta\rangle_{norm}$ for the first six values $n = 0, 1, \dots, 5$ and for the squeezing parameter $\zeta = (3 - \sqrt{5})/2 = 0.381966$ from a frog's perspective. The modulus $|\zeta|$ was chosen in such a way that the relation of the lengths of the major to the minor squeezing axes x_{max} and x_{min} which are given by

$$x_{max} = \sqrt{\hbar \frac{1 + |\zeta|}{1 - |\zeta|}}, \quad x_{min} = \sqrt{\hbar \frac{1 - |\zeta|}{1 + |\zeta|}}, \quad (4.9)$$

becomes equal to $\sqrt{5}$ (another reason for this choice was indicated in section 3). As the plot range is chosen, the maximal possible range from $-1/\pi$ to $1/\pi$ which can be taken on by the normalized Wigner quasiprobability $W(q, p)$. The maximal values $+1/\pi$ or $-1/\pi$ are reached at the coordinate origin $q = 0, p = 0$ for states with even or odd parity, respectively, or in more general cases with displacements for states with definite displaced parity correspondingly defined with respect to this displacement. The Wigner quasiprobability $W(q, p)$ of the states $|0, n; \zeta\rangle_{norm}$ for the first six values $n = 0, 1, \dots, 5$ and for the squeezing parameter $\zeta = 0.5$ from a bird's perspective are given in [23].

5. Photon statistics of the squeezed-state excitations

The photon statistics is described by the photon distribution function p_m which is expressed in terms of the matrix elements $\langle m|\beta, n; \zeta\rangle$ and taking into account $|m\rangle = |0, m; 0\rangle$ by

$$p_m = N_n^2(|\zeta|) |\langle 0, m; 0|\beta, n; \zeta\rangle|^2. \quad (5.1)$$

The corresponding matrix elements can be taken either in closed form (3.2), (3.5) or as series from Eq.(3.6) as the following special case

$$\begin{aligned} & \langle 0, m; 0|\beta, n; \zeta\rangle \\ = & (1 + |\zeta|^2)^{\frac{1}{4}} \exp \left\{ -\frac{(\beta + \zeta\beta^*)\beta^*}{2} \right\} \frac{(-1)^n}{\sqrt{m!n!}} \sum_{j=0}^{\{m,n\}} \frac{(-1)^j m!n!}{j!(m-j)!(n-j)!} (\sqrt{1 + |\zeta|^2})^j \\ & \left(\frac{\sqrt{2\zeta}}{2} \right)^{m-j} \left(\frac{\sqrt{2\zeta^*}}{2} \right)^{n-j} H_{m-j} \left(\frac{\beta + \zeta\beta^*}{\sqrt{2\zeta}} \right) H_{n-j} \left(\sqrt{\frac{1 + |\zeta|^2}{2\zeta^*}} \beta^* \right), \end{aligned} \quad (5.2)$$

whereas the normalization constant $N_n(|\zeta|)$ is given by Eqs.(2.9) and (2.11).

Let us consider some special cases. For vanishing squeezing parameter ζ , one obtains from $|\beta, n; \zeta\rangle$ the displaced Fock states $|\beta, n; 0\rangle \equiv |\beta, n\rangle$ already in the normalized form (see Eq.(2.6)) and from Eq.(5.2) it follows with the asymptotic expressions for the Hermite functions $H_n(z) \rightarrow (2z)^n$

$$\begin{aligned} \langle 0, m; 0|\beta, n; 0\rangle &= \exp \left(-\frac{|\beta|^2}{2} \right) \frac{(-1)^n}{\sqrt{m!n!}} \sum_{j=0}^{\{m,n\}} \frac{(-1)^j m!n!}{j!(m-j)!(n-j)!} \beta^{m-j} \beta^{*n-j} \\ &= \exp \left(-\frac{|\beta|^2}{2} \right) \sqrt{\frac{n!}{m!}} \beta^{m-n} L_n^{m-n}(|\beta|^2) \\ &= \exp \left(-\frac{|\beta|^2}{2} \right) \sqrt{\frac{m!}{n!}} (-\beta^*)^{n-m} L_m^{n-m}(|\beta|^2), \quad N_n^2(0) = 1, \end{aligned} \quad (5.3)$$

where $L_n^\nu(z)$ denotes the associated Laguerre polynomials (cf., e.g., [9]). The photon statistics of displaced Fock states $|\beta; n\rangle$ for $n = 0, 1, \dots, 5$ and $n = 5, 6, \dots, 10$ and for the squeezing parameter $|\beta| = \sqrt{25/2} = 3.53533$ is shown in Figs.3 and 4. These figures

demonstrate the influence of the excitation on the photon distribution structure. As we see, for larger excitations the structure becomes more homogeneous.

In the special case of squeezed coherent states $|\beta, 0; \zeta\rangle$, one obtains from Eq.(5.2)

$$\begin{aligned}\langle 0, m; 0 | \beta, 0; \zeta \rangle &= (1 + |\zeta|^2)^{\frac{1}{4}} \exp \left\{ -\frac{(\beta + \zeta\beta^*)\beta^*}{2} \right\} \frac{1}{\sqrt{m!}} \left(\frac{\sqrt{2}\zeta}{2} \right)^m H_m \left(\frac{\beta + \zeta\beta^*}{\sqrt{2}\zeta} \right), \\ N_0^2(|\zeta|) &= \left(\frac{1 - |\zeta|^2}{1 + |\zeta|^2} \right)^{\frac{1}{2}}.\end{aligned}\quad (5.4)$$

In particular, one obtains for squeezed vacuum states $|0, 0; \zeta\rangle$

$$\begin{aligned}\langle 0, 2m; 0 | 0, 0; \zeta \rangle &= (1 + |\zeta|^2)^{\frac{1}{4}} \frac{\sqrt{2m!}}{2^m m!} (-\zeta)^m, \quad \langle 0, 2m+1; 0 | 0, 0; \zeta \rangle = 0, \\ N_0^2(|\zeta|) &= \left(\frac{1 - |\zeta|^2}{1 + |\zeta|^2} \right)^{\frac{1}{2}}.\end{aligned}\quad (5.5)$$

The squeezed vacuum states $|0, 0; \zeta\rangle$ are superpositions of even Fock states $|2m\rangle$. The states $|0, 1; \zeta\rangle$ are squeezed Fock states $|1\rangle$ for which one obtains

$$\begin{aligned}\langle 0, 2m+1; 0 | 0, 1; \zeta \rangle &= (1 + |\zeta|^2)^{\frac{3}{4}} \frac{\sqrt{(2m+1)!}}{2^m m!} (-\zeta)^m, \quad \langle 0, 2m; 0 | 0, 1; \zeta \rangle = 0, \\ N_1^2(|\zeta|) &= \left(\frac{1 - |\zeta|^2}{1 + |\zeta|^2} \right)^{\frac{3}{2}}.\end{aligned}\quad (5.6)$$

These are superpositions of odd Fock states $|2m+1\rangle$. More generally, one obtains for the states $|0, n; \zeta\rangle$

$$\begin{aligned}&\langle 0, n+2m; 0 | 0, n; \zeta \rangle \\ &= \left(\sqrt{1 + |\zeta|^2} \right)^{n+\frac{1}{2}} \frac{\sqrt{(n+2m)!n!}}{2^m (n+m)!} (-\zeta)^m \sum_{l=0}^{[n/2]} \frac{(n+m)!}{l!(l+m)!(n-2l)!} \left(\frac{\zeta + \zeta^*}{4(1 + |\zeta|^2)} \right)^l \\ &= (1 + |\zeta|^2)^{\frac{1}{4}} \frac{\sqrt{(n+2m)!n!}}{2^m (n+m)!} (-\zeta)^m P_n^{(m,m)}(\sqrt{1 + |\zeta|^2}), \quad m \geq -\left[\frac{n}{2} \right], \\ &\langle 0, n+1+2m; 0 | 0, n; \zeta \rangle = 0, \quad m \geq -\left[\frac{n+1}{2} \right], \\ &N_n^2(|\zeta|) = \sqrt{\frac{1 - |\zeta|^2}{1 + |\zeta|^2}} \left(P_n \left(\frac{1 + |\zeta|^2}{1 - |\zeta|^2} \right) \right)^{-1},\end{aligned}\quad (5.7)$$

where $[\nu]$ denotes the integer part of ν . The states $|0, n; \zeta\rangle$ are only for $n = 0$ and $n = 1$ squeezed Fock states which can be obtained by applying a unitary squeezing operator onto the Fock states $|0\rangle$ and $|1\rangle$, respectively, but they cannot be obtained

for $n \geq 2$ by applying a unitary squeezing operator to the corresponding Fock states $|n\rangle$. This is due to the property that the unitary squeezing operators involves squares of the annihilation operator, i.e. a^2 , in the exponent which applied to the Fock states $|n\rangle$ with $n \geq 2$, do not annihilate these Fock states and the unitary and the nonunitary approach become inequivalent. It is, however, possible to determine initial states from which $|0, n; \zeta\rangle$ can be obtained by the same unitary squeezing operators as in the cases $n = 0$ and $n = 1$ from $|0\rangle$ and $|1\rangle$ (see, [9]) or one can first act with equivalent unitary or nonunitary squeezing operators onto the vacuum state and have then to take the corresponding excitation operators as in the given definition of the states.

Figures 5, 6, and 7 show the photon statistics of the normalized states $|\beta, n; \zeta\rangle$ for $n = 0$ and $|\beta| = 5$ in dependence on the squeezing parameter ζ in case that the large axis of the squeezing ellipse is perpendicular to the displacement parameter β ($\beta/\sqrt{\zeta}$ real numbers, Figs.5 and 6) and parallel to the displacement parameter β ($\beta/\sqrt{\zeta}$ imaginary numbers, Fig.7). The first special case will be called “perpendicular squeezing” ($\beta/\sqrt{\zeta}$ real numbers) and the second special case “parallel squeezing” ($\beta/\sqrt{\zeta}$ imaginary numbers) or we will speak about “perpendicular geometry” and “parallel geometry,” respectively. Conditionally, one can also speak about “amplitude squeezing” and “phase squeezing,” respectively, but this is only correct in a certain approximation if the quasidistributions are essentially restricted to small sectors of the phase plane because amplitude squeezing is squeezing in radial direction and phase squeezing is squeezing in angular direction. The characterization of the considered special cases by the kind of complex numbers $\beta/\sqrt{\zeta}$ is an invariant characterization meaning that it is independent of the chosen position of the coordinate system (q, p) in the phase plane [44]. In case of perpendicular geometry, we see the oscillations of the Fock-state occupation after the main occupation with pulse-like shape for large squeezing parameters $|\zeta| \approx 0.75 \dots 1$. This range of squeezing parameters is amplified in Fig.6. The oscillations can be explained by the neighbourhood to zeroes of the Hermite polynomials for real arguments. Figures 5 and 7 show for vanishing squeezing parameter ζ the photon statistics of coherent states (Poissonian statistics). Figures 8 and 9 are the analogous pictures for $n = 10$ and displacement parameter $|\beta| = \sqrt{15} = 3.87298$. These photon statistics also show oscillatory behaviour where in the second case for an intermediate range of values of the squeezing parameter $|\zeta| \approx 0.25 \dots 0.5$ some Fock states $|n\rangle$ with low numbers n are occupied and after a gap in the distribution appears again a region of occupation with an pulse-like shape. The cooperation of displacement, squeezing, and excitation leads to a diversity of possible features in the photon statistics that must be studied furthermore.

6. Expectation values for the squeezed-state excitations

The expectation value of an arbitrary operator A for the squeezed-state excitations $|\beta, n; \zeta\rangle$ is given by

$$\overline{A} \equiv \frac{\langle \beta, n; \zeta | A | \beta, n; \zeta \rangle}{\langle \beta, n; \zeta | \beta, n; \zeta \rangle} = \frac{\langle 0, n; \zeta | (D(\beta, \beta^*))^\dagger A D(\beta, \beta^*) | 0, n; \zeta \rangle}{\langle 0, n; \zeta | 0, n; \zeta \rangle}. \quad (6.1)$$

The scalar product in the denominator determines the normalization factor of the states $|\beta, n; \zeta\rangle$ and is expressed by special Jacobi or Legendre polynomials in Eq.(2.11). A possible approach for the calculation of expectation values of ordered powers of the annihilation and creation operator, i.e. ordered moments, is the following. First we calculate the expectation value of the displacement operator $D(\alpha, \alpha^*)$. Due to

$$(D(\beta, \beta^*))^\dagger D(\alpha, \alpha^*) D(\beta, \beta^*) = \exp(\alpha\beta^* - \alpha^*\beta) D(\alpha, \alpha^*), \quad (6.2)$$

one finds

$$\overline{D(\alpha, \alpha^*)} = \exp(\alpha\beta^* - \alpha^*\beta) \frac{\langle 0, n; \zeta | \alpha, n; \zeta \rangle}{\langle 0, n; \zeta | 0, n; \zeta \rangle}, \quad (6.3)$$

where the scalar product $\langle 0, n; \zeta | \alpha, n; \zeta \rangle$ can be taken from Eq.(3.7) by setting $m = n$ and by substituting $\beta \rightarrow \alpha$. For the well-known identities (e.g., [13])

$$\begin{aligned} \exp(\alpha a^\dagger) \exp(-\alpha^* a) &= \exp\left(\frac{\alpha\alpha^*}{2}\right) D(\alpha, \alpha^*), \\ \exp(-\alpha^* a) \exp(\alpha a^\dagger) &= \exp\left(-\frac{\alpha\alpha^*}{2}\right) D(\alpha, \alpha^*), \end{aligned} \quad (6.4)$$

one has

$$\begin{aligned} a^{\dagger l} a^k &= (-1)^k \left\{ \frac{\partial^{k+l}}{\partial \alpha^{*k} \partial \alpha^l} \exp\left(\frac{\alpha\alpha^*}{2}\right) D(\alpha, \alpha^*) \right\}_{\alpha=\alpha^*=0}, \\ a^k a^{\dagger l} &= (-1)^k \left\{ \frac{\partial^{k+l}}{\partial \alpha^{*k} \partial \alpha^l} \exp\left(-\frac{\alpha\alpha^*}{2}\right) D(\alpha, \alpha^*) \right\}_{\alpha=\alpha^*=0}. \end{aligned} \quad (6.5)$$

Hence we find the normally and antinormally ordered moments for the normalized states $|\beta, n; \zeta\rangle_{norm}$ by

$$\begin{aligned}\overline{a^{\dagger l} a^k} &= \frac{1}{\langle 0, n; \zeta | 0, n; \zeta \rangle} \\ &\quad \left\{ \left(\beta - \frac{\partial}{\partial \alpha^*} \right)^k \left(\beta^* + \frac{\partial}{\partial \alpha} \right)^l \exp \left(\frac{\alpha \alpha^*}{2} \right) \langle 0, n; \zeta | \alpha, n; \zeta \rangle \right\}_{\alpha=\alpha^*=0}, \\ \overline{a^k a^{\dagger l}} &= \frac{1}{\langle 0, n; \zeta | 0, n; \zeta \rangle} \\ &\quad \left\{ \left(\beta - \frac{\partial}{\partial \alpha^*} \right)^k \left(\beta^* + \frac{\partial}{\partial \alpha} \right)^l \exp \left(-\frac{\alpha \alpha^*}{2} \right) \langle 0, n; \zeta | \alpha, n; \zeta \rangle \right\}_{\alpha=\alpha^*=0},\end{aligned}\quad (6.6)$$

where we have separated the displacement part $\exp(\alpha\beta^* - \alpha^*\beta)$ from the differentiations and obtain in such a way by applying the binomial formula the moments as sums over powers of the displacement parameters β and β^* . However, it is not easy to calculate the necessary derivatives in these expressions for arbitrary numbers k and l . They can be expressed by multivariable Hermite polynomials but the transition to usual polynomials is then very complicated. Therefore, we represent here a second approach which gives the possibility to calculate these ordered moments for low order in a more direct way.

In [23], was derived that the operators $a(\zeta)$ and $(a(-\zeta))^\dagger$ defined by

$$a(\zeta) \equiv \frac{a + \zeta a^\dagger}{\sqrt{1 + \zeta \zeta^*}}, \quad (a(-\zeta))^\dagger \equiv \frac{a^\dagger - \zeta^* a}{\sqrt{1 + \zeta \zeta^*}}, \quad [a(\zeta), (a(-\zeta))^\dagger] = I, \quad (6.7)$$

play the role of the annihilation and creation operator of the states $|0, n; \zeta\rangle$ (see Eq.(6.17) in [23]). With the decomposition

$$a = \frac{a(\zeta) - \zeta(a(-\zeta))^\dagger}{\sqrt{1 + \zeta \zeta^*}}, \quad a^\dagger = \frac{(a(-\zeta))^\dagger + \zeta^* a(\zeta)}{\sqrt{1 + \zeta \zeta^*}}, \quad (6.8)$$

one obtains the action of the operators a and a^\dagger onto the states $|0, n; \zeta\rangle$ in the form

$$\begin{aligned}a|0, n; \zeta\rangle &= \frac{\sqrt{n}|0, n-1; \zeta\rangle - \zeta\sqrt{n+1}|0, n+1; \zeta\rangle}{\sqrt{1 + \zeta \zeta^*}}, \\ a^\dagger|0, n; \zeta\rangle &= \frac{\sqrt{n+1}|0, n+1; \zeta\rangle + \zeta^*\sqrt{n}|0, n-1; \zeta\rangle}{\sqrt{1 + \zeta \zeta^*}},\end{aligned}\quad (6.9)$$

and as a consequence, for example,

$$\begin{aligned}
a^\dagger a |0, n; \zeta\rangle &= \frac{1}{1 + \zeta \zeta^*} \left\{ (n - (n+1)\zeta \zeta^*) |0, n; \zeta\rangle \right. \\
&\quad \left. - \zeta \sqrt{(n+2)(n+1)} |0, n+2; \zeta\rangle + \zeta^* \sqrt{n(n-1)} |0, n-2; \zeta\rangle \right\}, \\
aa^\dagger |0, n; \zeta\rangle &= \frac{1}{1 + \zeta \zeta^*} \left\{ (n+1 - n\zeta \zeta^*) |0, n; \zeta\rangle \right. \\
&\quad \left. - \zeta \sqrt{(n+2)(n+1)} |0, n+2; \zeta\rangle + \zeta^* \sqrt{n(n-1)} |0, n-2; \zeta\rangle \right\}. \quad (6.10)
\end{aligned}$$

With

$$(D(\beta, \beta^*))^\dagger a D(\beta, \beta^*) = a + \beta I, \quad (D(\beta, \beta^*))^\dagger a^\dagger D(\beta, \beta^*) = a^\dagger + \beta^* I, \quad (6.11)$$

and with the vanishing of the scalar products of the states $|0, n; \zeta\rangle$ and $|0, m; \zeta\rangle$ if the difference $|n - m|$ is an odd number, one easily finds

$$\bar{a} = \beta, \quad \overline{a^\dagger} = \beta^*. \quad (6.12)$$

In the same way, by using Eq.(6.10) and the explicit expressions for the scalar products of states $|0, n; \zeta\rangle$ and $|0, m; \zeta\rangle$ (see Eqs.(3.11) and (3.12)) one finds the expectation value of the number operator $N \equiv a^\dagger a$ in the form

$$\begin{aligned}
\overline{N} &\equiv \frac{\langle \beta, n; \zeta | a^\dagger a | \beta, n; \zeta \rangle}{\langle \beta, n; \zeta | \beta, n; \zeta \rangle} \\
&= \frac{\langle 0, n; \zeta | (a^\dagger + \beta^* I)(a + \beta I) | 0, n; \zeta \rangle}{\langle 0, n; \zeta | 0, n; \zeta \rangle} \\
&= \frac{1}{(1 + |\zeta|^2) P_n^{(0,0)} \left(\frac{1 + |\zeta|^2}{1 - |\zeta|^2} \right)} \left\{ (n - (n+1)|\zeta|^2) P_n^{(0,0)} \left(\frac{1 + |\zeta|^2}{1 - |\zeta|^2} \right) \right. \\
&\quad \left. + \frac{|\zeta|^2}{1 - |\zeta|^2} \left((n+2) P_n^{(1,1)} \left(\frac{1 + |\zeta|^2}{1 - |\zeta|^2} \right) - n P_{n-2}^{(1,1)} \left(\frac{1 + |\zeta|^2}{1 - |\zeta|^2} \right) \right) \right\} + \beta \beta^*. \quad (6.13)
\end{aligned}$$

With the following identity for Jacobi polynomials

$$(n+2) P_n^{(1,1)}(z) - n P_{n-2}^{(1,1)}(z) = 2(2n+1) P_n^{(0,0)}(z), \quad (6.14)$$

which can be proved by using the explicit representations of the Jacobi polynomials with equal upper indices (e.g., Eq.(A.3) in Appendix A), one obtains from Eq.(6.13) the following simple result of the expectation value of the number operator $N \equiv a^\dagger a$ for the normalized states $|\beta, n; \zeta\rangle_{norm}$

$$\overline{N} = \frac{n + (n+1)|\zeta|^2}{1 - |\zeta|^2} + |\beta|^2. \quad (6.15)$$

In an analogous way, one obtains from

$$\begin{aligned}
a^2|0, n; \zeta\rangle &= \frac{1}{1 + \zeta\zeta^*} \left\{ \sqrt{n(n-1)}|0, n-2; \zeta\rangle \right. \\
&\quad \left. - \zeta(2n+1)|0, n; \zeta\rangle + \zeta^2\sqrt{(n+2)(n+1)}|0, n+2; \zeta\rangle \right\}, \\
a^{\dagger 2}|0, n; \zeta\rangle &= \frac{1}{1 + \zeta\zeta^*} \left\{ \sqrt{(n+2)(n+1)}|0, n+2; \zeta\rangle \right. \\
&\quad \left. + \zeta^*(2n+1)|0, n; \zeta\rangle + \zeta^{*2}\sqrt{n(n-1)}|0, n-2; \zeta\rangle \right\}, \tag{6.16}
\end{aligned}$$

by using the representation of the scalar products by means of the Jacobi polynomials and with the identity in Eq.(6.14)

$$\begin{aligned}
\overline{a^2} &\equiv \frac{\langle \beta, n; \zeta | a^2 | \beta, n; \zeta \rangle}{\langle \beta, n; \zeta | \beta, n; \zeta \rangle} \\
&= -\frac{\zeta}{1 - |\zeta|^2} \left(2n + 1 + n \frac{P_{n-2}^{(1,1)}\left(\frac{1 + |\zeta|^2}{1 - |\zeta|^2}\right)}{P_n^{(0,0)}\left(\frac{1 + |\zeta|^2}{1 - |\zeta|^2}\right)} \right) + \beta^2, \quad \overline{a^{\dagger 2}} = \overline{a^2}^*. \tag{6.17}
\end{aligned}$$

The series of functions $f_n(y)$ defined by

$$f_n(y) \equiv \frac{P_{n-2}^{(1,1)}\left(\frac{1+y}{1-y}\right)}{P_n^{(0,0)}\left(\frac{1+y}{1-y}\right)} \tag{6.18}$$

has for the first six values of n the following form

$$\begin{aligned}
f_0(y) &= -(1-y), \quad f_1(y) = 0, \quad f_2(y) = \frac{(1-y)^2}{1+4y+y^2}, \quad f_3(y) = \frac{2(1-y)^2}{1+8y+y^2}, \\
f_4(y) &= \frac{3(1-y)^2(1+3y+y^2)}{1+16y+36y^2+16y^3+y^4}, \quad f_5(y) = \frac{4(1-y)^2(1+5y+y^2)}{1+24y+76y^2+24y^3+y^4}, \tag{6.19}
\end{aligned}$$

and gives a contribution in Eq.(6.17) only for $n \geq 2$. The uncertainty of the expectation values of the canonical operators Q and P is given by

$$\begin{aligned}
\overline{(\Delta Q)^2} &\equiv \overline{(Q - \overline{Q}I)^2} = \frac{\hbar}{2} \left\{ (2n+1) \frac{|1-\zeta|^2}{1-|\zeta|^2} - \frac{\zeta + \zeta^*}{1-|\zeta|^2} n f_n(|\zeta|^2) \right\}, \\
\overline{(\Delta P)^2} &\equiv \overline{(P - \overline{P}I)^2} = \frac{\hbar}{2} \left\{ (2n+1) \frac{|1+\zeta|^2}{1-|\zeta|^2} + \frac{\zeta + \zeta^*}{1-|\zeta|^2} n f_n(|\zeta|^2) \right\}, \tag{6.20}
\end{aligned}$$

and the corresponding symmetrized uncertainty correlation by

$$\overline{\Delta Q \Delta P + \Delta P \Delta Q} = \hbar \frac{i(\zeta - \zeta^*)}{1 - |\zeta|^2} (2n + 1 + n f_n(|\zeta|^2)). \quad (6.21)$$

Now it is easy to calculate from (6.20) the uncertainty sum and the uncertainty product. The uncertainty sum (or total uncertainty)

$$\overline{(\Delta Q)^2} + \overline{(\Delta P)^2} = \hbar(2n + 1) \frac{1 + |\zeta|^2}{1 - |\zeta|^2} \geq 2 \sqrt{\overline{(\Delta Q)^2} \overline{(\Delta P)^2}} \quad (6.22)$$

is invariant with respect to rotations of the system, i.e. depends not on the phase of squeezing parameter ζ but only on its modulus $|\zeta|$ and increases linearly with the excitation number n for fixed $|\zeta|$. The inequality is simply the relation between arithmetic and geometric means. The uncertainty product

$$\begin{aligned} \overline{(\Delta Q)^2} \overline{(\Delta P)^2} = \frac{\hbar^2}{4} \left\{ (2n + 1)^2 \frac{|1 - \zeta^2|^2}{(1 - |\zeta|^2)^2} \right. \\ \left. - n f_n(|\zeta|^2) (2(2n + 1) + n f_n(|\zeta|^2)) \frac{|\zeta + \zeta^*|^2}{(1 - |\zeta|^2)^2} \right\}, \quad (6.23) \end{aligned}$$

is not rotation-invariant and depends in a complicated manner on the modulus and on the phase of the squeezing parameter ζ and increases mainly with the excitation number. One can analyse Eqs.(6.20) by a computer for low values of the excitation number n and finds that each factor $\overline{(\Delta Q)^2}$ and $\overline{(\Delta P)^2}$ can be reduced below the value $\hbar/2$ for coherent states and reaches for real (and only real) positive or negative values of ζ , in the limiting case $\zeta \rightarrow \pm 1$, the value zero for $\overline{(\Delta Q)^2}$ or $\overline{(\Delta P)^2}$, respectively. Contrary to this, the symmetrized uncertainty correlation according to Eq.(6.21) becomes extremal for pure imaginary values of the squeezing parameter ζ under fixed excitation number n .

In an analogous way, one can calculate the expectation values, for example, of the operators $a^{\dagger 2} a^2$, $a^2 a^{\dagger 2}$ or N^2 . However, the expressions obtained are very complicated and one cannot find identities for the Jacobi polynomials in such a way that the expressions in the numerator and in the denominator shortens in an essential way. We do not write down these complicated expressions but it is possible to find the effective polynomials in the numerator and denominator for low values of n in the states $|\beta, n; \zeta\rangle$ by using a computer.

7. Conclusion

We discussed statistical properties of the squeezed-state excitations both for the quadrature component of the light and for the photon number. As it turned out the Wigner quasiprobability and the photon distribution function for these states are described in terms of multivariable Hermite polynomials. The corresponding limiting cases of these multivariable Hermite polynomials expressed in terms of Legendre and Laguerre polynomials (and also Gegenbauer and Jacobi polynomials) gave a possibility both to calculate explicitly some series for usual Hermite polynomials and to express the normalization constant of the squeezed-state excitations and their scalar product in terms of the well-studied classical polynomials. Correspondingly, the coherent-state quasiprobability (Husimi–Kano function) was expressed in closed form in terms of one-variable Hermite polynomials and Legendre polynomials. The influence of displacement parameter and squeezing parameter onto the photon statistics in the states was calculated.

It turned out that there exists the range of these parameters for which the photon distribution function demonstrates highly oscillatory behaviour which is a characteristics of strongly nonclassical states known for squeezed states, even and odd coherent states, and correlated states. The dependence of the average photon number on the squeezing and displacement parameter in the squeezed-state excitations was explicitly given in terms of Jacobi polynomials. The main results of the work are given in the formulae (2.11), (3.15), (4.3), (6.15), and (6.20) in which in closed form are described quadrature and photon statistics of the squeezed-state excitations. Using the method of constructing the simple superposition states like even and odd coherent states [3] (or Schrödinger cat states [4]) it is possible to introduce the superposition states in which the partners of the superposition are the excitations of the squeezed coherent states introduced in [23, 24] and discussed in the present paper.

One could conclude that, being dependent on the physical parameters like squeezing parameter, displacement parameter, and excitation number parameter, the squeezed-state excitations demonstrate a reach range of possibilities to influence the photon statistics and promise to be realized experimentally in future as other members of the nonclassical-state family.

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Appendix A:

New polynomials and their relation to Jacobi, Gegenbauer, and Legendre polynomials

By comparison of two different methods of calculation of scalar products of the states in Eq.(2.12) for the special case of equal displacement parameter we obtained the following relation of new polynomials to the Jacobi polynomials $P_n^{(i,j)}(z)$ (see, e.g., [34]) with equal upper indices $i = j$ and with transformed argument z as well as multiplied by functions of z

$$\sum_{l=0}^{[n/2]} \frac{(-1)^l (n+j)!}{l!(l+j)!(n-2l)!} \left(\frac{x}{4}\right)^l = \left(\sqrt{1+x}\right)^n P_n^{(j,j)}\left(\frac{1}{\sqrt{1+x}}\right), \quad (\text{A.1})$$

where, according to the parity properties of the Jacobi polynomials, both signs of the square root $\sqrt{1+x}$ are possible, but one has to take the same signs of this square root in the argument of the Jacobi polynomials and in the multipliers. Before proving this relation, we make some transformations. With the substitution

$$x \equiv -\frac{4y}{(1+y)^2}, \quad \leftrightarrow \quad \sqrt{1+x} \equiv \frac{1-y}{1+y}, \quad \leftrightarrow \quad y \equiv -\frac{\sqrt{1+x}-1}{\sqrt{1+x}+1}, \quad (\text{A.2})$$

one brings Eq.(A.1) into the following form in which we apply it in this paper

$$\sum_{l=0}^{[n/2]} \frac{(n+j)!}{l!(l+j)!(n-2l)!} \left(\frac{y}{(1+y)^2}\right)^l = \left(\frac{1-y}{1+y}\right)^n P_n^{(j,j)}\left(\frac{1+y}{1-y}\right). \quad (\text{A.3})$$

Now, by the substitution

$$y \equiv \frac{z-1}{z+1}, \quad \leftrightarrow \quad z \equiv \frac{1+y}{1-y}, \quad (\text{A.4})$$

one finds from Eq.(A.3)

$$P_n^{(j,j)}(z) = z^n \sum_{l=0}^{[n/2]} \frac{(n+j)!}{l!(l+j)!(n-2l)!} \left(\frac{z^2-1}{4z^2}\right)^l. \quad (\text{A.5})$$

The following relation of the Jacobi polynomials with equal upper indices $P_n^{(j,j)}(z)$ to the Gegenbauer (or Ultraspherical) polynomials $C_n^\lambda(z)$ is known ([34], chap.10.9, Eq.(4))

$$\begin{aligned} P_n^{(j,j)}(z) &= \frac{(n+j)!(2j)!}{(n+2j)!j!} C_n^{j+\frac{1}{2}}(z) \\ &= \frac{(n+j)!(2j)!}{(n+2j)!j!\Gamma(j+\frac{1}{2})} \sum_{k=0}^{[n/2]} \frac{(-1)^k \Gamma(n-k+j+\frac{1}{2})}{k!(n-2k)!} (2z)^{n-2k}, \end{aligned} \quad (\text{A.6})$$

where additionally the explicit representation of the Gegenbauer polynomials is inserted. There exists another relation between the Jacobi polynomials with equal upper indices $P_l^{(j,j)}(z)$ and the associated Legendre polynomials $P_n^j(z)$ of the following kind [11]

$$P_n^{(j,j)}(z) = \frac{(n+j)!}{(n+2j)!} \left(\frac{2}{\sqrt{1-z^2}} \right)^j P_{n+j}^j(z). \quad (\text{A.7})$$

However, this relation is uniquely determined only in case of even $j = 2m$ for the two possible signs of the square root $\sqrt{1-z^2}$ and one must be cautious when applying this relation. The reason is that the associated Legendre polynomials $P_l^j(z)$ with odd upper indices j after the substitution $z \equiv \cos(\theta)$ do not only depend on $\cos(\theta)$ but also depend on $\sin(\theta)$. In the particular case $j = 0$, one obtains from Eqs.(A.6) and (A.7)

$$P_n^{(0,0)}(z) = P_n(z) = C_n^{\frac{1}{2}}(z), \quad (P_n(z) \equiv P_n^0(z)). \quad (\text{A.8})$$

This is one case where the relation to the Legendre polynomials is uniquely defined.

Let us now prove the given relations. First we prove the transition from the sum representation of $P_n^{(j,j)}(z)$ in Eq.(A.5) to the sum representation in Eq.(A.6). By applying the binomial formula to $(1 - 1/z^2)$ in Eq.(A.5), one finds

$$P_n^{(j,j)}(z) = \frac{(n+j)!}{(n+2j)!} \sum_{k=0}^{[n/2]} \frac{(-1)^k}{k!} z^{n-2k} \sum_{l=k}^{[n/2]} \frac{(n+2j)!}{(l+j)!(l-k)!(n-2l)!2^{2l}}. \quad (\text{A.9})$$

The interior sum over l can be evaluated as follows

$$\sum_{l=k}^{[n/2]} \frac{(n+2j)!}{(l+j)!(l-k)!(n-2l)!2^{2l}} = \frac{2^{n-2k}(2j)!\Gamma(n-k+j+\frac{1}{2})}{j!\Gamma(j+\frac{1}{2})(n-2k)!}. \quad (\text{A.10})$$

This can be proved by complete induction from $n \rightarrow n+1$ since the result is obviously true for $n = 0$ and all integer j and $k \geq 0$. With the decomposition of the factor $n+1+2j = (n+1-2l) + 2(l+j)$ in the numerator and the substitution $l \rightarrow l+1$ in the second arising sum term, the proof is easily to perform. Thus, it is proved that the expression denoted by $P_n^{(j,j)}(z)$ in Eq.(A.5) has the representation by the Gegenbauer polynomials given in Eq.(A.6) but it is not yet proved by this approach

that it is identical with the Jacobi polynomial denoted by the same symbol. However, the relations of the Gegenbauer polynomials to special Jacobi polynomials are known as already mentioned. Nevertheless, it is instructive to follow a direct transition from the explicit general representation of the Jacobi polynomials specialized for equal upper indices to the explicit representation by the Gegenbauer polynomials in Eq.(A.6). We make this next.

Starting from the explicit representation of the Jacobi polynomials with equal upper indices by applying the binomial formula and the convolution formula for the binomial coefficients (e.g., [9], Eqs.(6.6) and (6.7)), one quickly proceeds to the following double sum

$$\begin{aligned}
P_n^{(j,j)}(z) &= \frac{1}{2^n} \sum_{m=0}^n \frac{(n+j)!^2}{m!(n-m)!(n+j-m)!(j+m)!} \\
&\quad \frac{1}{2} \left\{ (z-1)^{n-m}(z+1)^m + (z+1)^{n-m}(z-1)^m \right\} \\
&= \frac{(n+j)!^2}{(n+2j)!2^n} \sum_{k=0}^{[n/2]} z^{n-2k} \frac{(2n-2k+2j)!}{(n-2k)!} \\
&\quad \sum_{r=0}^{\{2k,n+j\}} \frac{(-1)^r}{r!(2k-r)!(n+j-r)!(n+j-2k+r)!}. \tag{A.11}
\end{aligned}$$

The interior sum can be evaluated in the following way (see also [49], chap.4.2.5, Eq.(29))

$$\begin{aligned}
\sum_{r=0}^{\{2k,l\}} \frac{(-1)^r}{r!(2k-r)!(l-r)!(l-2k+r)!} &= \frac{1}{l!(2l-2k)!} \sum_{s=0}^{\{2k,l\}} \frac{(-2)^s(2l-s)!}{s!(2k-s)!(l-s)!} \\
&= \frac{(-1)^k}{k!l!(l-k)!}. \tag{A.12}
\end{aligned}$$

Using this sum evaluation with the substitution $l = n+j$ in Eq.(A.11) together with the “doubling” formula for Gamma functions, one obtains $P_n^{(j,j)}(z)$ in the form of Eq.(A.6). The sum evaluation in Eq.(A.12) can be proved by complete induction from $k \rightarrow k+1$ since it is true for $k=0$. The sum is transformed in Eq.(A.12) first into another sum. This transformation can be made by setting $(-1)^r = (1-2)^r$ and by applying the binomial formula to these powers and then after changing the order of summations the convolution formula for the binomial coefficients to reduce the double sum to a simple sum. It seems that in this transformed form the proof by complete induction is easier to perform by multiplying the relation with $(2k+2)(2k+1)$ and by using the decomposition $(2k+2)(2k+1) = (2k+2-s)(2k+1-s) + 2s(2k+2-s) + s(s-1)$. In the general case of the Jacobi polynomials $P_n^{(i,j)}(z)$ with unequal upper indices $i \neq j$, one can make analogous expansions in powers of z (all powers z^{n-l} and not only z^{n-2k} are present in this case), where the coefficients are given by finite sums but it was not

to see how these sums can be evaluated by a simple expression as it was possible and demonstrated for $i = j$. This is an advantage of the direct derivation.

Last we consider the relation in Eq.(A.7). The Associated Legendre polynomials are defined by [48]

$$\begin{aligned} P_l^j(z) &\equiv \frac{(-1)^l}{2^l l!} (\sqrt{1-z^2})^j \frac{d^{l+j}}{dz^{l+j}} (1-z^2)^l \\ &= (\sqrt{1-z^2})^j \frac{1}{2^l} \sum_{k=0}^{[(l-j)/2]} \frac{(-1)^k (2l-2k)!}{k! (l-j-2k)! (l-k)!} z^{l-j-2k}. \end{aligned} \quad (\text{A.13})$$

With the substitution $l = n + j$ and by applying the doubling formula for the Gamma function, one finds

$$P_{n+j}^j(z) = (\sqrt{1-z^2})^j \frac{2^j}{\sqrt{\pi}} \sum_{k=0}^{[n/2]} \frac{(-1)^k \Gamma(n-k+j+\frac{1}{2})}{k! (n-2k)!} (2z)^{n-2k}. \quad (\text{A.14})$$

The comparison of this formula with Eq.(A.6) by applying the doubling formula of the Gamma function for $(2j)!$ reveals the identity in Eq.(A.7) with the discussed problems of the sign of the square root.

Despite the close relations of the polynomials in Eq.(A.1) to Jacobi polynomials with equal upper indices, one must look to them as to new original polynomials because these relations include nonlinear argument transformations. It seems to be interesting to get more relations for these new polynomials as, for example, generating functions, recurrence relations, weight factors, and integrals.

Appendix B:

Multivariable Hermite polynomials

In the Appendix, we review the properties of two-dimensional and multidimensional Hermite polynomials following [35], [36], [50].

The two-dimensional Hermite polynomials $H_{mn}^{\{R\}}(y_1, y_2)$ are defined by means of the generating function [34]

$$\exp \left\{ -\frac{1}{2} \mathbf{a} R \mathbf{a} + \mathbf{a} R \mathbf{y} \right\} = \sum_{m,n=0}^{\infty} \frac{a_1^m a_2^n}{m!n!} H_{mn}^{\{R\}}(\mathbf{y}). \quad (\text{B.1})$$

Here a_1 and a_2 are arbitrary complex numbers combined into the two-dimensional vector $\mathbf{a} = (a_1, a_2)$:

$$\mathbf{a} R \mathbf{a} = \sum_{i,k=0}^2 a_i R_{ik} a_k, \quad \mathbf{a} R \mathbf{y} = \sum_{i,k=0}^2 a_i R_{ik} y_k, \quad (\text{B.2})$$

and R is the symmetric 2×2 -matrix

$$R = \begin{pmatrix} R_{11} & R_{12} \\ R_{12} & R_{22} \end{pmatrix}. \quad (\text{B.3})$$

Introducing the notation

$$r \equiv \frac{R_{12}}{\sqrt{R_{11} R_{22}}}, \quad (\text{B.4})$$

one can obtain [35] the following formula for the two-dimensional Hermite polynomial of zero arguments:

$$H_{mn}^{\{R\}}(0, 0) = \mu_{mn}! (-1)^{\frac{m+n}{2}} R_{11}^{m/2} R_{22}^{n/2} (r^2 - 1)^{\frac{m+n}{4}} P_{(m+n)/2}^{|m-n|/2} \left(\frac{r}{\sqrt{r^2 - 1}} \right), \quad (\text{B.5})$$

where

$$\mu_{mn} = \min(m, n), \quad (\text{B.6})$$

and integers m, n must have the same parity (otherwise the right-hand side equals zero). Here the Hermite polynomial is expressed in terms of associated Legendre function.

For coinciding indices, we get

$$H_{nn}^{\{R\}}(0, 0) = n!(-\det R)^{\frac{n}{2}} P_n \left(\frac{-R_{12}}{\sqrt{-\det R}} \right), \quad (\text{B.7})$$

$P_n(z)$ being the usual Legendre polynomial.

Equation (B.5) can be rewritten in the following equivalent forms, using the Jacobi polynomials

$$H_{mn}^{\{R\}}(0, 0) = \frac{m!n!(-1)^{\frac{m+n}{2}}}{2^{\frac{|m-n|}{2}} \left(\frac{m+n}{2}\right)!} \left(R_{11}^m R_{22}^n (r^2 - 1)^{\mu_{mn}}\right)^{\frac{1}{2}} P_{\mu_{mn}}^{\left(\frac{|m-n|}{2}, \frac{|m-n|}{2}\right)} \left(\frac{r}{\sqrt{r^2 - 1}}\right), \quad (\text{B.8})$$

or using the Gegenbauer polynomials

$$H_{mn}^{\{R\}}(0, 0) = \frac{\mu_{mn}!|m-n|!(-1)^{\frac{m+n}{2}}}{2^{\frac{|m-n|}{2}} \left(\frac{|m-n|}{2}\right)!} \left(R_{11}^m R_{22}^n (r^2 - 1)^{\mu_{mn}}\right)^{\frac{1}{2}} C_{\mu_{mn}}^{\frac{|m-n|+1}{2}} \left(\frac{r}{\sqrt{r^2 - 1}}\right). \quad (\text{B.9})$$

For nonzero vector \mathbf{y} , function $H_{mn}^{\{R\}}(y_1, y_2)$ can be written as a finite sum of products of the usual Hermite polynomials,

$$\begin{aligned} & \left(\frac{R_{11}^m R_{22}^n}{2^{m+n}}\right)^{-\frac{1}{2}} H_{mn}^{\{R\}}(y_1, y_2) \\ &= \sum_{j=0}^{\mu_{mn}} \left(-\frac{2R_{12}}{\sqrt{R_{11}R_{22}}}\right)^j \frac{m!n!}{j!(m-j)!(n-j)!} H_{m-j} \left(\frac{\zeta_1}{\sqrt{2R_{11}}}\right) H_{n-j} \left(\frac{\zeta_2}{\sqrt{2R_{22}}}\right), \end{aligned} \quad (\text{B.10})$$

where

$$\zeta_1 = R_{11}y_1 + R_{12}y_2, \quad \zeta_2 = R_{12}y_1 + R_{22}y_2. \quad (\text{B.11})$$

Using Pauli matrix

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (\text{B.12})$$

one has for nonzero arguments the relation of the Hermite polynomial to the associated Laguerre polynomial

$$H_{mn}^{\{t\sigma_x\}}(y_1, y_2) = \mu_{mn}! t^{\nu_{mn}} (-1)^{\mu_{mn}} y_1^{\frac{(n-m+|n-m|)}{2}} y_2^{\frac{(m-n+|m-n|)}{2}} L_{\mu_{mn}}^{|m-n|}(ty_1y_2), \quad (\text{B.13})$$

where

$$\nu_{mn} = \max(m, n). \quad (\text{B.14})$$

For zero argument, we have the formula

$$H_{mn}^{\{R\}}(0, 0) = \left(\frac{R_{11}^m R_{22}^n}{2^{m+n}}\right)^{\frac{1}{2}} \sum_{l=0}^{\left[\frac{\mu_{mn}}{2}\right]} \frac{(-1)^{\frac{m+n}{2}} m!n!}{l! \left(l + \frac{|m-n|}{2}\right)! (\mu_{mn} - 2l)!} (2r)^{\mu_{mn} - 2l}, \quad (\text{B.15})$$

where m and n must have the same parity.

For multivariable Hermite polynomials one can calculate some integrals. Thus, if we denote

$$\mathbf{m} = m_1, m_2, \dots, m_N, \quad \mathbf{n} = n_1, n_2, \dots, n_N, \quad m_i, n_i = 0, 1, \dots, \quad (\text{B.16})$$

one has

$$\int d\mathbf{x} H_{\mathbf{m}}^{\{S\}}(\mathbf{x}) H_{\mathbf{n}}^{\{T\}}(\Lambda\mathbf{x} + \mathbf{d}) \exp(-\mathbf{x}M\mathbf{x} + \mathbf{c}\mathbf{x}) = \frac{\pi^{N/2}}{\sqrt{\det M}} \exp\left(\frac{1}{4}\mathbf{c}M^{-1}\mathbf{c}\right) H_{\mathbf{m}\mathbf{n}}^{\{R\}}(\mathbf{y}), \quad (\text{B.17})$$

where the symmetric $2N \times 2N$ -matrix

$$R = \begin{pmatrix} R_{11} & R_{12} \\ \tilde{R}_{12} & R_{22} \end{pmatrix} \quad (\text{B.18})$$

with $N \times N$ -blocks R_{11} , R_{22} , R_{12} is expressed in terms of symmetric $N \times N$ -matrices S , T , M and $N \times N$ -matrix Λ in the form

$$\begin{aligned} R_{11} &= S - \frac{1}{2}SM^{-1}S, \\ R_{22} &= T - \frac{1}{2}T\Lambda M^{-1}\tilde{\Lambda}T, \\ \tilde{R}_{12} &= -\frac{1}{2}T\Lambda M^{-1}S. \end{aligned} \quad (\text{B.19})$$

Here the matrix $\tilde{\Lambda}$ is transposed matrix Λ , and \tilde{R}_{12} is transposed matrix R_{12} . The $2N$ -vector \mathbf{y} is expressed in terms of N -vectors \mathbf{c} and \mathbf{d} in the form

$$\mathbf{y} = R^{-1} \begin{pmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{pmatrix}, \quad (\text{B.20})$$

where N -vectors \mathbf{z}_1 and \mathbf{z}_2 are

$$\begin{aligned} \mathbf{z}_1 &= \frac{1}{4}(SM^{-1} + M^{-1}S)\mathbf{c} \\ \mathbf{z}_2 &= \frac{1}{4}(T\Lambda M^{-1} + M^{-1}\tilde{\Lambda}T)\mathbf{c} + T\mathbf{d}. \end{aligned} \quad (\text{B.21})$$

For matrices $S = 2$, $T = 2$, the above formula (B.17) yields

$$\begin{aligned} &\int d\mathbf{x} \left\{ \prod_{i=1}^N H_{m_i}(x_i) H_{n_i} \left(\sum_{k=1}^N \Lambda_{ik} x_k + d_i \right) \right\} \exp(-\mathbf{x}M\mathbf{x} + \mathbf{c}\mathbf{x}) \\ &= \frac{\pi^{N/2}}{\sqrt{\det M}} \exp\left(\frac{1}{4}\mathbf{c}M^{-1}\mathbf{c}\right) H_{\mathbf{m}\mathbf{n}}^{\{R\}}(\mathbf{y}), \end{aligned} \quad (\text{B.22})$$

with $N \times N$ -blocks R_{11} , R_{22} , R_{12} expressed in terms of $N \times N$ -matrices M and Λ in the form

$$\begin{aligned} R_{11} &= 2(1 - M^{-1}), \\ R_{22} &= 2(1 - \Lambda M^{-1} \tilde{\Lambda}), \\ \tilde{R}_{12} &= -2\Lambda M^{-1}. \end{aligned} \quad (\text{B.23})$$

The $2N$ -vector \mathbf{y} is expressed in terms of N -vectors \mathbf{c} and \mathbf{d} in the form of Eq.(B.20) with

$$\begin{aligned} \mathbf{z}_1 &= M^{-1} \mathbf{c}, \\ \mathbf{z}_2 &= \frac{1}{2}(\Lambda M^{-1} + M^{-1} \tilde{\Lambda}) \mathbf{c} + 2\mathbf{d}, \end{aligned} \quad (\text{B.24})$$

according to Eq.(B.21).

In the special case $N = 1$, the matrices Λ and M become scalars and one finds from Eq.(B.22)

$$\int_{-\infty}^{+\infty} dx H_m(x) H_n(\Lambda x + d) \exp(-Mx^2 + cx) = \sqrt{\frac{\pi}{M}} \exp\left(\frac{c^2}{4M}\right) H_{mn}^{\{R\}}(y_1, y_2), \quad (\text{B.25})$$

with

$$R = 2 \begin{pmatrix} 1 - \frac{1}{M} & -\frac{\Lambda}{M} \\ -\frac{\Lambda}{M} & 1 - \frac{\Lambda^2}{M} \end{pmatrix}, \quad (\text{B.26})$$

and

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \frac{c}{M} \\ \frac{\Lambda c}{M} + 2d \end{pmatrix}, \quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{1}{2\left(1 - \frac{1+\Lambda^2}{M}\right)} \begin{pmatrix} \frac{c}{M} + \frac{2\Lambda d}{M} \\ \frac{\Lambda c}{M} - 2\left(1 - \frac{1}{M}\right)d \end{pmatrix}. \quad (\text{B.27})$$

The parameters ζ_1 and ζ_2 in the arguments of the usual Hermite polynomials in the representation of the two-variable Hermite polynomials according to Eqs.(B.10) and (B.11) are then

$$\zeta_1 = z_1, \quad \zeta_2 = z_2. \quad (\text{B.28})$$

Thus, we obtain the following explicit representation of this integral by the usual Hermite polynomials

$$\begin{aligned} & \sqrt{\frac{M}{\pi}} \exp\left(-\frac{c^2}{4M}\right) \int_{-\infty}^{+\infty} dx H_m(x) H_n(\Lambda x + d) \exp(-Mx^2 + cx) \\ &= \left(\sqrt{\frac{M-1}{M}}\right)^m \left(\sqrt{\frac{M-\Lambda^2}{M}}\right)^n \sum_{j=0}^{\mu_{mn}} \frac{m!n!}{j!(m-j)!(n-j)!} \left(\frac{2\Lambda}{\sqrt{(M-1)(M-\Lambda^2)}}\right)^j \\ & \quad H_{m-j}\left(\frac{c}{2\sqrt{M(M-1)}}\right) H_{n-j}\left(\frac{\Lambda c + 2d}{2\sqrt{M(M-\Lambda^2)}}\right). \end{aligned} \quad (\text{B.29})$$

This means that in the considered special case the calculation of z_1 , z_2 , and R is sufficient for the representation of the result of the integral in Eq.(B.25) by sums over the usual Hermite polynomials whereas the additional calculation of y_1 and y_2 is necessary for the representation by the two-variable Hermite polynomials.

Bibliography

- [1] D.F. Walls, **306**, 141 (1983).
- [2] J.N. Hollenhorst, Phys. Rev. D **19**, 1669 (1969).
- [3] V.V. Dodonov, I.A. Malkin, and V.I. Man'ko, Physica **72**, 597 (1974).
- [4] B. Yurke and D. Stoler, Phys. Rev. Lett. **57**, 13 (1986).
- [5] V. Bužek, G. Adam, and G. Drobný, Annals Phys. (New York) **245**, 37 (1996).
- [6] K.E. Cahill and R.J. Glauber, Phys. Rev. **177**, 1857; 1882 (1969).
- [7] M. Boiteux and A. Levelut, J. Phys. A: Math. Nucl. Gen. **6**, 589 (1963).
- [8] F.A.M. de Oliveira, M.S. Kim, P.L. Knight, and V. Bužek, Phys. Rev. A **41**, 2645 (1990).
- [9] A. Wünsche, Quant. Opt. **3**, 359 (1991).
- [10] P. Král, J. Mod. Opt. **37**, 889 (1990).
- [11] G. Schrader, V.M. Akulin, V.I. Man'ko, and W. Schleich, Phys. Rev. **48**, 2398 (1993).
- [12] R.J. Glauber, Phys. Rev. Lett. **10**, 84 (1963).
- [13] R.J. Glauber, Phys. Rev **131**, 2766 (1963).
- [14] E.C.G. Sudarshan, Phys. Rev. Lett. **10**, 277 (1963).
- [15] J.R. Klauder, J. Math. Phys. **4**, 1055 (1963).
- [16] V. Bužek and P.L. Knight, *Progress in Optics*, ed. E. Wolf (Elsevier, Amsterdam, 1995), Vol. 34, p. 1.
- [17] W. Schleich and J.A. Wheeler, Nature **326**, 574 (1987).
- [18] A. Vourdas and R.M. Weiner, Phys. Rev. A **36**, 5866 (1987).

- [19] G.S. Agarwal and G. Adam, Phys. Rev. A **39**, 6259 (1989).
- [20] V.V. Dodonov, E.V. Kurmyshev, and V.I. Man'ko, Phys. Lett. A **79**, 150 (1980).
- [21] V.V. Dodonov, A.B. Klimov, and V.I. Man'ko, Proceedings of the Lebedev Physical Institute, **200**, 56 (Nauka, Moscow, 1991).
- [22] E.C.G. Sudarshan, Ch.B. Chiu, and G. Bhamathi, Phys. Rev. A **52**, 43 (1995).
- [23] A. Wünsche, Quant. Semicl. Opt. **8**, 343 (1996).
- [24] A. Wünsche, Fourth International Conference on Squeezed States and Uncertainty Relations, Taiyuan, China, 1995, Nasa Conference Publication, Greenbelt, Maryland, 1996.
- [25] Z. Bialynicka-Birula, Phys. Rev. **173**, 1207 (1968).
- [26] U.M. Titulaer and R.J. Glauber, Phys. Rev. **145**, 1041 (1966).
- [27] M.M. Nieto and D.R. Truax, Phys. Rev. Lett. **71**, 2843 (1993).
- [28] V. Spiridonov, Phys. Rev. A **52**, 1909 (1995).
- [29] C. K. Law, Phys. Rev. A **49**, 433 (1994).
- [30] V. I. Man'ko, in: *Symmetries in Physics*, eds. A. Frank and K.-B. Wolf (Springer-Verlag, Berlin, 1992), p. 311.
- [31] V. V. Dodonov, A. B. Klimov, and V. I. Man'ko, Phys. Lett. A **49**, 255 (1990).
- [32] V. V. Dodonov, A. B. Klimov, and D. E. Nikonov, J. Math. Phys. **34**, 2742 (1993).
- [33] R. Jáuregui, C. Villarreal, and S. Hacyan, Mod. Phys. Lett. A **10**, 619 (1995); Phys. Rev. A **52**, 601 (1995).
- [34] H. Bateman and A. Erdélyi, *Higher Transcendental Functions* (Mc Graw-Hill, New York, 1953), Vol. 2.
- [35] V.V. Dodonov and V.I. Man'ko, J. Math. Phys. **35**, 4277 (1994).
- [36] V.V. Dodonov, J. Phys. A **27**, 6191 (1994).
- [37] V. V. Dodonov, V. I. Man'ko, and V. V. Semjonov, Nuovo Cim. B **83**, 145 (1984).
- [38] M. Kauderer, J. Math. Phys. **34**, 4221 (1993).
- [39] I. A. Malkin, V. I. Man'ko, and D. A. Trifonov, J. Math. Phys. **14**, 576 (1973).
- [40] K. Vogel, V.M. Akulin, and W.P. Schleich, Phys. Rev. Lett. **71**, 1816 (1993).

- [41] A. Wünsche, Appl. Phys. B **60**, 119 (1995).
- [42] N. Lütkenhaus and S.M. Barnett, Phys. Rev. A **51**, 3340 (1995).
- [43] A. Wünsche, Ann. Phys. (Leipzig) **1**, 181 (1992).
- [44] A. Wünsche, in: *Second International Workshop on Squeezed States and Uncertainty Relations, Moscow, 1992*, Nasa Conference Publication, **3219** (Greenbelt, Maryland, 1993), p. 217.
- [45] E. Wigner, Phys. Rev. **40**, 749 (1932).
- [46] K. Husimi, Proc. Phys. Math. Soc. Japan **22**, 264 (1940).
- [47] Y. Kano, J. Math. Phys. **6**, 1913 (1965).
- [48] L.D. Landau and E.M. Lifshitz, *Quantum Mechanics* (Pergamon Press, New York, 1958).
- [49] A.P. Prudnikov, Yu.A. Brychkov, and O.I. Marichev, *Integrals and Series. Vol. 1. Elementary Functions* (Gordon and Breach, New York, 1986).
- [50] V. V. Dodonov and V. I. Man'ko, *Invariants and Evolution of Nonstationary Quantum Systems*, Proceedings of the Lebedev Physical Institute, **183** (Nova Science, New York, 1989).

Figure captions

Fig.1

Coherent-state quasiprobability $Q(q, p)$ of the squeezed-state excitations with the parameters $\beta = 0$, $\zeta = 0.381966$, and $n = 0, 1, 2, 3, 4, 5$ corresponding to a), b), c), d), e), f) from a bird's perspective ($\hbar = 1$).

Fig.2

Wigner quasiprobability $W(q, p)$ of the squeezed-state excitations with the same parameters $\beta = 0$, $\zeta = 0.381966$, and $n = 0, 1, 2, 3, 4, 5$ corresponding to a), b), c), d), e), f) as in Fig.1 from a frog's perspective ($\hbar = 1$).

Fig.3

Photon distribution function for $\beta = 3.53533$, $\zeta = 0$ and for excitation numbers $n = 0, 1, \dots, 5$, i.e., for displaced Fock states.

Fig.4

Photon distribution function for $\beta = 3.53533$, $\zeta = 0$ and for excitation numbers $n = 5, 6, \dots, 10$, i.e., for displaced Fock states.

Fig.5

Photon distribution function for $\beta = 5$, $n = 0$ as the function of the squeezing parameter ζ , $0 \leq \zeta \leq 1$, i.e., for squeezed coherent states and perpendicular geometry (large squeezing axis perpendicular to displacement).

Fig.6

Photon distribution function for $\beta = 5$, $n = 0$ as the function of the squeezing parameter ζ , $0.75 \leq \zeta \leq 1$, i.e., for squeezed coherent states and perpendicular geometry and amplification of the range from 0.75 to 1.

Fig.7

Photon distribution function for $\beta = 5$, $n = 0$ as the function of the squeezing parameter ζ , $0 \leq -\zeta \leq 1$, i.e., for squeezed coherent states and parallel geometry (large squeezing axis parallel to displacement).

Fig.8

Photon distribution function for $\beta = 3.87298$, $n = 10$ as the function of the squeezing parameter ζ , $0 \leq \zeta \leq 1$, i.e., for squeezed-state excitations with perpendicular geometry.

Fig.9

Photon distribution function for $\beta = 3.87298$, $n = 10$ as the function of the squeezing parameter ζ , $0 \leq -\zeta \leq 1$, i.e., for squeezed-state excitations with parallel geometry.